Scaling analysis of Langevin-type equations

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The approach of scaling behavior of open dissipative systems, which was proposed by Hentschel and Family [Phys. Rev. Lett. **66**, 1982 (1991)], is developed to analyze several models. The results show there are two scaling regions, a strong-coupling region and a weak-coupling region, in each model. The dynamic renormalization-group results are exactly the same as the results in the weak-coupling region. The scaling exponents in the strong-coupling region and the crossover behavior are also discussed.

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The scaling behavior of a growing interface is a challenging problem of both theoretical and practical interest [1–3]. One approach to studying this behavior is by deriving Langevin-type equations, which are assumed to incorporate physics. Typical of such equations is the Kardar-Parisi-Zhang (KPZ) equation [4] for the height fluctuations $h(r,t)$ in an interface growing with a velocity λ normal to the interface,

$$\frac{\partial h}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta(r,t)\ , \tag{1}$$

where

$$\langle \eta(r,t)\eta(r',t')\rangle = 2D\delta(r-r')\delta(t-t')\ . \tag{2}$$

Other examples are molecular-beam-epitaxy (MBE) growth [5] and the Sun, Guo, and Grant (SGG) equation [6] for the interface height of a driven interface with conservation law.

These nonlinear equations are, in general, insoluable. Therefore, most of the efforts in the past have been focused on determining the scaling behavior of the fluctuations using a dynamic renormalization-group (RG) approach or by direct numerical solutions of these equations [4,5,7,8]. The dynamic RG has had only a limited success in the study of dissipative systems because there is no Hamiltonian formulation for nonequilibrium processes, and in most cases, RG equations cannot be formulated or solved. Numerical solutions, on the other hand, are of practical importance, but can only give approximate values of the scaling exponents. But since numerical results are only approximate and cannot be used to determine universality and crossover behavior, they do not provide physical insight into these processes. In addition, unlike the scaling behavior at a critical point which is described by a single exponent, fluctuations in dynamical systems can have different scaling behaviors depending on the length scale. Thus it would be useful to have an approach that could be readily applied to Langevin-type equations and which could be used to determine the exponents in any dimension for different scaling regions.

Recently, Hentschel and Family have proposed a very interesting approach for studying the scaling behavior of Langevin-type equations for dissipative dynamical systems [9]. Their approach is similar in spirit to the scaling arguments used by Kolmogorov in the analysis of fully developed turbulence and is based on the analogy between Langevin-type equations and the forced Navier-Stokes equations. They show that the approach can be applied to derive not only the critical exponents in any dimension, but also the fluctuation amplitudes, critical dimensions, and regions of validity, where various exponents may be observed. The approach may be considered a nonequilibrium equivalent of the Flory theory for equilibrium scaling. They demonstrate the approach by several models, such as interface growth with the KPZ equation, surface growth with conservation law, self-organized criticality, and surface growth with quenched randomness.

The physics of this approach is as follows: when coarse grained over length scales l, each separate term in the equation must be of the same order of magnitude or negligible to show dynamic scaling. Only under these circumstances can scaling behavior arise. The validity of a scaling region can then be found in a self-consistent manner from the region of length scales over which intrinsic assumptions apply. The art, as in Flory theory, lies in estimating the magnitude of individual terms, especially as, in general, we are dealing with self-affine and anisotropic systems which introduce several length scales into the estimate.

In this Brief Report, we analyze the scaling behavior of MBE growth, interface growth with conservation law, and the interface growth KPZ equation with spatial and temporal correlations, respectively. We show that there are two regions in each model, the strong-coupling region and weak-coupling region. The dynamic RG results are exactly the weak-coupling results. The scaling exponents of the two regions are equal when $d<d_b$; d_b is the bifurcation dimensionality, and as $d>d_b$, they will become branched.

Molecular-beam epitaxy. The first open system we examine is the technologically important MBE-growth process under ideal MBE-growth conditions [5], i.e., atom stochastic growth without any bulk defects or surface overhangs. The growth must obey a mass conservation law, and is described by

$$\frac{\partial h}{\partial t}=-\nu\nabla^4 h+\frac{\lambda}{2}\nabla^2(\nabla h)^2+\eta(r,t)\ . \tag{3}$$

The correlations satisfy Eq. (2). The results of dynamic RG and various numerical simulations are consistent with each other. The physically interesting dimension for MBE growth is $d=2+1$ where we obtain $\alpha=\frac{2}{3}$, $\beta=\frac{1}{5}$, and $z=\frac{10}{3}$. The situation with $d=1+1$ is also physically realizable in the context of adatom motion on vicinal surfaces with steps and the corresponding exponents are $\alpha=1$, $\beta=\frac{1}{3}$, and $z=3$.

We assume that at long times $t \gg t_l$ and averaged over length scales l, the typical magnitude of the fluctuations in the interfacial height scale as $\langle [h(r+l,t) - h(r,t)]^2 \rangle \sim h_l^2$, and that at long times these fluctuation last for times of the order t_l. Then, apart from the noise, and averaged over scale l, the various terms in the growth equation may be estimated as $\langle \partial h/\partial t \rangle_l \sim h_l/t_l$, $\nu\langle \nabla^2 h \rangle_l \sim \nu h_l/l^2$, and $(\lambda/2)\langle (\nabla h)^2 \rangle_l \sim (\lambda/2)(h_l^2/l^2)$.

To proceed further we need to estimate the average noise on these length and time scales. For white noise we estimate its mean-square fluctuations on length scales l and times scales t_l as $\eta \sim [D/(S_l t_l)]^{1/2}$, where S_l is the average surface area of the interface on length scale l. This is a sample consequence of adding uncorrelated random variables. We estimate the surface area of the growth on length scale l as $S_l \sim (h_l^2+l^2)^{(d-1)/2}$ and, consequently, for smooth surfaces $\eta_l \sim [D/(l^{d-1}t_l)]^{1/2}$ while for rough surfaces $\eta_l \sim [D/(h^{d-1}t_l)]^{1/2}$.

To derive the rough and dynamic exponents, we assume that at sufficiently large length scales $l \gg l_{\text{in}}$ the nonlinear term in the MBE equation will dominate the surface diffusion. The region where this assumption is valid is defined by $h_l \gg \nu/\lambda$. Equating the $\partial h/\partial t$ term with the nonlinear term implies that a typical fluctuation lasts for times $t_l \sim l^4/(\lambda h_l)$ and the scaling behavior of these two terms implies $\alpha+z=4$. Equating our estimate for the noise fluctuation in a rough interface $h_l \gg l$ (a condition yielding an outer length scale l_{out}) to the inertial term then yields

$$h_l \sim (D/\lambda)^{1/(d+2)} l^{4/(d+2)} \tag{4}$$

and, consequently, $\alpha=4/(d+2)$ in this regime. The inner length scale l_{in} can now be found by inserting Eq. (4) into the self-consistency condition. We can find the scaling behavior of h_t with time t at short times by reexpressing h_t in terms of t_l and assuming scaling is valid for $t \ll t_l$ with the result

$$h_t \sim D^{1/(d+1)} t_l^{1/(d+1)} \tag{5}$$

and, therefore, $\beta=1/(d+1)$.

The outer length scale can be found by substituting Eq. (4) into the criterion for the existence of a rough interface and this implies that we may expect to observe the exponents only in models in the strong-coupling limit where the dimensionless parameter $g=\lambda^{d-1}D/\nu^d \gg 1$, which is analogous to the Reynolds member describing hydrodynamic turbulence.

If $g \ll 1$, it is also possible to find a region where the coupling is weak and can be treated as a perturbation. We estimate that the noise term is for the smooth surface $\eta_l \sim [D/(l^{d-1}t_l)]^{1/2}$. Equating the $\partial h/\partial t$ term with the nonlinear term also gets $t_l \sim l^4/\lambda h_l$. Equating our estimate for the noise fluctuation in a smooth interface yields

$$h_l \sim (D/\lambda)^{1/3} l^{(5-d)/3} \tag{6}$$

and, consequently, $\alpha=(5-d)/3$ while

$$h_t \sim D^{4/(7+d)} \lambda^{(1-d)/(7+d)} t^{(5-d)/(7+d)} \tag{7}$$

and, thus, $\beta=(5-d)/(7+d)$.

The exponents are exactly the same as the results derived by Lai and Das Sarma [5] using the dynamic RG method. This shows that dynamic RG results are in the smooth region which satisfies $g \ll 1$, i.e., in the weak-coupling region. When comparing the two sets of exponents, we find that when $d<2$, they are equal. As $d>2$, they will become branched. The strong-coupling surface is rougher than that of the weak-coupling surface.

Interface growth with a conservation law (SGG equation). In order to study the effect of conservation law on interface growth, SGG uses the dynamic RG to study the nonlinear Langevin equation [6],

$$\frac{\partial h}{\partial t} = -\nabla^2 \left[\nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 \right] + \eta(r,t) , \tag{8}$$

where

$$\langle \eta(r,t)\eta(r',t') \rangle = -2D\nabla^2 \delta(r-r')\delta(t-t') . \tag{9}$$

Again neglecting the diffusion term as small as large enough length scales and equating our estimate for the time variation in height fluctuations $\langle \partial h/\partial t \rangle_l \sim h_l/l_t$ to our estimate for the nonlinear term $(\lambda/2)\langle \nabla^2(\nabla h)^2 \rangle_l \sim (\lambda/2)(h_l^2/l^4)$ yields the identity $\alpha+z=4$.

In the weak-coupling region $g \ll 1$, equating the $\partial h/\partial t$ term to the smooth surface noise estimate $\eta \sim [D/(l^{d+1}t_l)]^{1/2}$ yields

$$h_l \sim (D/\lambda)^{1/3} l^{(3-d)/3} , \tag{10}$$

and, consequently, $\alpha=(3-d)/3$ with $d_c=3$ and

$$t_l \sim (D\lambda^2)^{-1/3} l^{(9+d)/3} . \tag{11}$$

Consequently, $z=(9+d)/3$ in this region is again the exact dynamic RG perturbation result of SGG. In the strong-coupling regime $g \gg 1$, equating the $\partial h/\partial t$ term to the rough surface noise estimate $\eta_l \sim [D/(h^{d-1}l^2 t_l)]^{1/2}$ yields

$$h_l \sim (D/\lambda)^{1/(d+2)} l^{2/(d+2)} \tag{12}$$

and, consequently, $\alpha=2/(d+2)$ while

$$t_l \sim D^{-1/(d+2)} \lambda^{-(d+1)/(d+2)} l^{(4d+6)/(d+2)} , \tag{13}$$

and, consequently, $z=(4d+6)/(d+2)$. These two set of exponents are equal at $d=1$, while $d=2$ is the bifurcation point.

Interface growth KPZ equation with special correlations. Medina *et al.* [7] and Zhang [10] consider a generalization of the KPZ equation for interfacial growth in which the noise, instead of being a δ function, has become the spatial correlation

$$\langle \eta(r,t)\eta(r',t') \rangle = D'|r-r'|^{2\rho-(d-1)} \delta(t-t') . \tag{14}$$

We expect the effect of long-range correlations in the noise to change our estimate of the noise fluctuation average over length scales l and time scales t_l. When $g \gg 1$, we estimate the noise as $\eta_l \sim [D/(h^{-2\rho+(d-1)}t_l)]^{1/2}$. As all other relationships remain unchanged, the behavior of h_l and h_t can immediately be found with the result

$$h_l \sim (D'/\lambda)^{1/(2+d-2\rho)} l^{2/(2+d-2\rho)} \tag{15}$$

and, therefore, $\alpha = 2/(2+d-2\rho)$ while

$$h_t \sim (D't)^{1/(1+d-2\rho)} . \tag{16}$$

Thus, $\beta = 1/(1+d-2\rho)$. This is the result of Hentschel and Family. It is a strong-coupling result. In the weak-coupling region, where $g \ll 1$, the noise estimate is the smooth region $\eta_l \sim [D/(l^{-2\rho+(d-1)} t_l)]^{1/2}$. In this condition, the results can be found as

$$h_l \sim (D'/\lambda)^{1/3} l^{(2-d'+2\rho)/3} \tag{17}$$

and, consequently, $\alpha = (2-d'+2\rho)/3$, while

$$t_l \sim D'^{1/3} \lambda^{-2/3} l^{(4+d'-2\rho)/3} \tag{18}$$

and thus $z = (4+d'-2\rho)/3$, where $d' = d-1$. This is exactly the perturbation results of Medina *et al.* [7].

KPZ equation with temporal correlation. Considering the interface growth with temporal correlations, the noise correlation instead of white noise is

$$\langle \eta(r,t)\eta(r',t') \rangle = 2D\delta^d(r-r')|t-t'|^{2\theta} . \tag{19}$$

In the weak-coupling region, $g \ll 1$, we estimate that the noise is in smooth region $\eta_l \sim [D/(l^{d-1} t_l^{-2\theta+1})]^{1/2}$, and the following result can be found:

$$h_l \sim D^{1/(2\theta+3)} \lambda^{-(2\theta+1)/(2\theta+3)} l^{(2-d+4\theta)/(3+2\theta)} \tag{20}$$

and, therefore, $\alpha = (2-d+4\theta)/(3+2\theta)$, while

$$t_l \sim D^{-1/(2\theta+3)} \lambda^{-2/(2\theta+3)} l^{(8-d)/(2\theta+3)} , \tag{21}$$

so $z = (8-d)/(2\theta+3)$. In the strong-coupling region, $g \ll 1$, we use the rough region noise $\eta_l \sim [D/(h^{d-1} t_l^{-2\theta+1})]^{1/2}$ which yields

$$h_l \sim D^{1/(2\theta+d+2)} \lambda^{-(2\theta+1)/(2\theta+d+2)} l^{(2+4\theta)/(2\theta+d+2)} \tag{22}$$

and, therefore, $\alpha = (2+4\theta)/(2\theta+d+2)$ while

$$t_l \sim D^{-1/(2\theta+d+2)} \lambda^{-(d+1)/(2\theta+d+2)} l^{(2d+2)/(2\theta+d+2)} , \tag{23}$$

and so $z = (2d+2)/(2\theta+d+2)$. Here we get the scaling exponents for the KPZ equation with temporal correlations in the strong-coupling and weak-coupling regions, respectively.

In conclusion, we have developed this approach to study the scaling behavior of fluctuation in dissipative dynamical systems, which is proposed by Hentschel and Family. We have illustrated that there are two different regions for obtaining scaling exponents, the strong-coupling rough region and weak-coupling smooth region. In particular, we show that the dynamic RG perturbation result is exactly the result in the weak-coupling scale region. In addition to distinguishing the two regions, we provide insights into the strong-coupling region that cannot be observed in the dynamic RG calculation. The different scale regions manifest themselves in regions where the noise terms are estimates in different forms. Clearly, the range of applicability of this approach will go beyond the above analysis.

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